Applications GeoGebra into Teaching Some Topics of Mathematics at the College Level

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Abstract. This paper presents new trends in technology and learning through Geogebra, which could be especially important for the future development of e-learning for College mathematics. Also, contribution of this paper is presentation of methodological frames on several specifics examples for teaching mathematics at the college level on interactive and creative way. Discussed in this paper are the findings of a didactic experiment where some number of undergraduate students were exposed to an innovative (to the departments) method of teaching and learning a part of calculus supported by GeoGebra. Statistical analysis confirmed the fact that the use of the applets created with the help of GeoGebra and used in differential calculus teaching had a positive effect on the understanding and knowledge of the students.

Keywords: GeoGebra, college mathematics, creative environment, didactic experiment.

1.   GEOGEBRA

At the moment, GeoGebra (from Geometry and Algebra) is one of the most innovative, open -code math software (GNU General Public License) which can be freely downloaded from www.geogebra.org. GeoGebra works on a wide spectrum of operating system platforms which have Java virtual machine installed on.

Markus Hohenwarter created free, open-source dynamic mathematics software GeoGebra, which is used for both teaching and learning mathematics from middle school through college to the University level (see Hohenwarter & Preiner, 2007). GeoGebra offers geometry, algebra and calculus features in a fully connected, compacted and easy-to-use software environment. In other words, this tool extends the concepts of dynamic geometry to the fields of algebra and mathematical analysis.

Designed specifically for educational purposes, GeoGebra can help students grasp experimental, problem-oriented and research-oriented learning of mathematics, both in the classroom and at home. Students can simultaneously use a computer algebra system and an interactive geometric

system; by doing this, they can increase their cognitive abilities in the best way.

GeoGebra is dynamic geometry software that supports constructions with points, lines and all conic sections. It also provides typical features for a Computer Algebra System such as finding important points of functions (roots, local extrema and inflection points of functions), direct input of equations and coordinates, finding derivates and integrals of the entered functions. That is the reason why GeoGebra is a good choice for multiple presentations of mathematical objects.

The basic idea of GeoGebra's interface is to provide two presentations of each mathematical object in its algebra and graphics windows. If you change an object in one of these windows, its presentation in the other one will be immediately updated. Computer algebra systems (such as Mathematica, Maple, and so on, e.g.) and dynamic geometry software (such as Geometer's Sketchpad, Cabri Geometry, and so on, e.g.) are powerful technological tools for teaching mathematics. Numerous research results suggest that these software packages can be used to encourage discovery, experimentation and visualization in traditional teaching of mathematics. However, researches suggest that, for the majority of teachers, the main problem is how to provide the technology necessary for the successful integration of technology into teaching (Ruthven & Hennessy, 2004).

So, the suggested solution for applying technology in the college math teaching is the software packet GeoGebra. The advantages of using GeoGebra are:

- In comparison to a graph calculator, GeoGebra is more user-friendly. GeoGebra offers easy-to-use interface, multilingual menus, commands and help.
- GeoGebra encourages students` projects in mathematics, multiple presentations and experimental and guided discovery learning.
- Students can personalize their own creations through the adaptation of interface (e.g. font size, language, quality of graphics, color, coordinates, line thickness, line style and other features).
- GeoGebra was created to help students gain a better understanding of mathematics. Students can manipulate variables easily by simply dragging "free" objects around the plane of drawing, or by using sliders. Students can generate changes using a technique of manipulating free objects, and then they can learn how the dependent objects will be affected. In this way, students have the opportunity to solve problems by investigating mathematical relations dynamically.
- Cooperative learning is the right context for a mathematics course (Dubinsky & Schwingendorf, 2004). Lecturing should be replaced by a task-oriented interactive classroom. The primary role of teaching is not to lecture, explain, or otherwise attempt to "transfer" mathematical knowledge, but to create situations for students that will foster their making the necessary mental constructions. In that sense, GeoGebra provides a good opportunity for cooperative learning, i.e. cooperative problem solving in

small groups, or whole class interactive teaching, or individual/group
student presentations.

- The algebra input allows the user to generate new objects or to modify
  those already existing, by the command line. The worksheet files can easily
  be published as Web pages.
- GeoGebra stimulates teachers to use and assess technology in:
  visualization of mathematics; investigations in mathematics; interactive
  mathematics classes on site or at a distance; mathematics and its
  applications, etc.

The deficiencies of using GeoGebra are:

- Students without previous programming experience will hardly enter
  algebraic commands in the input box. Although the basic commands are
  not difficult to learn, students may feel embarrassed or quite at a loss of
  what to do.
- Some methodological approaches (e.g. independent exploring and
  experimenting) can not be appropriate for many students.
- In a technical sense, GeoGebra does not have an in-built support for
  animation. So, including the modules for animating in GeoGebra should
  become an important technical element for future versions.
- Future extensions of the software GeoGebra will surely include more
  symbolic features of computer algebra systems which will further increase
  possible complex applications in the mathematical analyses, and 3D
  extensions.
- Limited research on the impact of GeoGebra on teaching and learning of
  mathematics.

GeoGebra has been rapidly gaining popularity among teachers and
researchers around the world, because it is easy-to-use dynamic mathematics
software that combines many aspects of different mathematical packages. In
addition, because of its open-source nature, an extensive user community has
developed around it.

2.   Math Visualization and Exploration by Using GeoGebra

The visualization that is possible with today's dynamic software enables the
student to see and explore mathematical relations and concepts that were
difficult to "show" in past prior to technology. The most meta-analysis of
research in the area of instructional technology in education show that
students who use technology in their learning had positive gains in learning
outcomes over students who learned without technology (Waxman, Connell,
Gray, 2002).

Although my teaching methods are still traditional (textbook, lecture notes,
lectures, assignments, final exam) my plan is to try to make some changes in
the way of work in order to make my teaching courses more effective and to

improve the learning outcomes of my students. The goal is to use GeoGebra to provide an environment for active exploration of mathematical structures through multiple representations, or to show students some aspects of the mathematics that are not possible with pen and paper.

2.1. Interactive Examples in Teaching and Learning Linear Algebra with GeoGebra

Using examples and questions given below, the teacher will guide students to fully understanding of the meaning and nature of solutions to linear systems of equations. These examples illustrate low-order systems to assist students in personalizing important concepts about systems of linear equations by "discovery method". The teacher should define the terms "consistent, inconsistent, homogeneous", "one solution, no solution, infinite solutions", and precise definitions of "linear equation, linear system, nonlinear system," and "solution". Also, the teacher should strongly encourage geometric visualizations through the lesson plan (Dikovic, 2007).

Example 1. Consider families of equations which take the form

$$ax + by = c. \tag{1}$$

For different values of a, b, c, the graphs of these equations are lines in a two dimensional coordinate system. Let the students choose any real numbers $a_1, b_1, c_1, a_2, b_2, c_2$ to define a system of two equations in two unknowns:

$$a_1 x + b_1 y = c_1, \tag{2}$$
$$a_2 x + b_2 y = c_2.$$

These graphs can easily be generated using GeoGebra. GeoGebra allows modifying these equations dynamically by moving the points or the sliders. In this way, students are able to investigate all possible situations including the interesting special cases of no solution and infinitely many solutions.

Q1. Can you change the numbers of the second equation so that the resulting system has
- No solution? (see figure 2)
- Exactly one solution? (see figure 1)
- Infinitely many solutions?
- If you do get one (infinite) solution for your final answer, is this system consistent or inconsistent? Would the equations be dependent or independent?
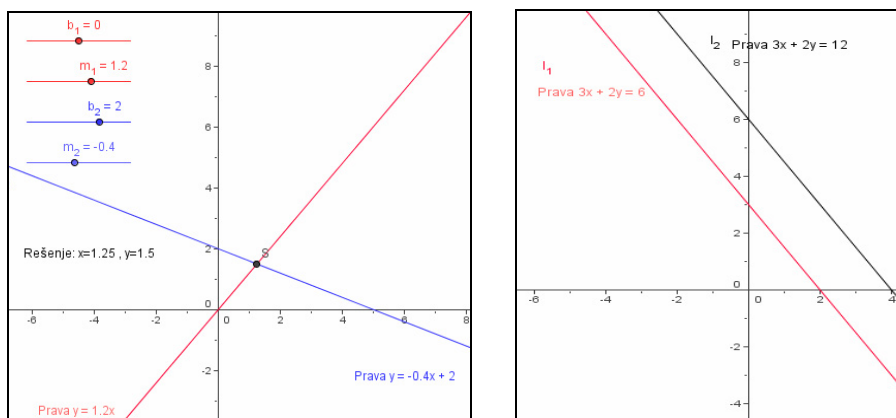
Fig. 1.   **Fig. 2.**

Q2. Student can set a_1, b_1, a_2, b_2 to values such that

$$a_1 \cdot b_2 - a_2 \cdot b_1 = 0 \qquad (3)$$

and consider the next questions: How are the two lines positioned with respect to each other? How many solutions the system has? What is the relationship between the slopes of the two lines?

Q3. Student can set all 6 constants to values such that

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \qquad (4)$$

and explain how many solutions the system has in that case.

Q4. Ask the students to randomly select numbers of a third equation and to graph it along with the first two. Normally such a random selection should generate an inconsistent system with a large probability, whose lines bound a triangular region. Does this new system have a solution and whether they have a solution or not?

Also, GeoGebra brings to students and teachers the opportunity to individualize learning - to generate illustrative examples, to follow interesting topics to the desired depth, to choose their own problems and appropriate tools for solving them. The students can concentrate on ideas instead of trying to get the arithmetic right in the solution of the some linear system. Additionally in examples of solving linear system, GeoGebra can stimulate the students' geometric intuition through interesting visualizations in 2D.

2.2. Interactive Examples in Teaching and Learning Calculus with GeoGebra

2.2.1. The Slope of the Tangent Line to a Curve and plotting the function of derivative

The following applet introduces the derivative as a slope function (see figure 3). A quadratic function in the form

$$f(x) = ax^2 + bx + c \tag{5}$$

and its first derivative are explored simultaneously in order to gain deep understanding of the concept of the derivative and also its graphical meaning.
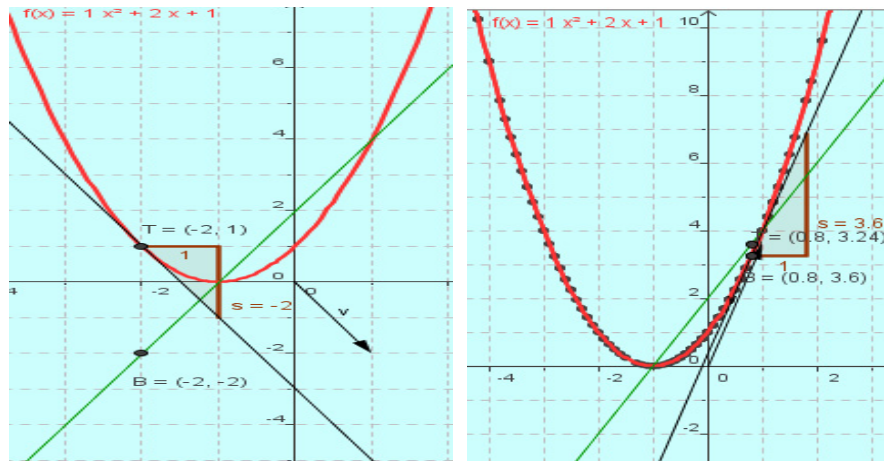

Fig. 3.

Students can use the sliders on the right side (a,b,c) to change the coefficients of quadratic equation, or slider (d) to change the position of the tangent line and observe as the graph of $f'(x)$ is drawn.

2.2.2. Using GeoGebra for Examining Continuity/Discontinuity of a Function

How does continuity affect limits? What is Cauchy .i.e. epsilon-delta definition of continuity?

Intuitively, a continuous function is a function for which, changes in some small neighborhood around input points, result in small changes in the output points. Because the symbolic relationships between limits and continuity are

quite abstract and complicated, it would be useful to start with the applet
designed to help users understand continuity in terms of limits, i.e. the
epsilon/delta definition of continuity at a point x_0.

In order to develop an intuitive understanding and discover of continuity,
next we can consider the following questions: What does it mean for a
function to be continuous? What do continuous functions look like?

– Example 1. *Let*

$$f(x) = \begin{cases} \dfrac{x-6}{x-3}, & x < 0 \\ 2, & x = 0 \\ \sqrt{4+x^2}, & x > 0 \end{cases} \tag{6}$$

See if the given function is continuous at the point x=0.

With the help of GeoGebra, an appropriate applet has been created, in
which the student gets the following algebraic explanation together with a
graphic one (see figure 4). The teacher can analyze and explain every
segment of the given algebraic explanation:

The function $f(x)$ is defined at $x = 0$, since $f(0) = 2$. The left-hand limit
of the function

$$\lim_{x->0_-} f(x) = \lim_{x->0_-} \frac{x-6}{x-3} = 2 \tag{7}$$

and the right-hand limit of the function

$$\lim_{x->0_+} f(x) = \lim_{x->0_+} \sqrt{4+x^2} = 2 \tag{8}$$

are calculated. As the left and the right limit values are equal, the conclusion
is that there is a limit value of the function

$$\lim_{x->0} f(x) = 2. \tag{9}$$

Since the condition that

$$\lim_{x->0} f(x) = 2 = f(0) = 2 \tag{10}$$

is fulfilled, the function $f(x)$ is continuous at $x = 0$.

On the graphic presentation, the applet also shows that f(x) is continuous
at $x = 0$. By moving the "movable" point A along the graph, and especially by
approaching the point $x = 0$ from the left and from the right, the student can

confirm that the function has no so-called "jumps" when it passes through the point $x = 0$. Graphically, applet shows that f(x) is continuous at $x = 0$. Student can move the point A, approaches to point at $x = 0$ and can verify that function has not "jumped" at $x = 0$.

Determine if the following function is continuous at x=0.

$$f(x)= \begin{cases} \frac{x-6}{x-3}, & x<0 \\ 2, & x=0 \\ \sqrt{4+x^2}, & x>0 \end{cases}$$

Function f is defined at x=0 since f(0)=2.

The left-hand limit $\quad \lim\limits_{x \to 0_-} f(x) = \lim\limits_{x \to 0_-} \frac{x-6}{x-3} = 2.$

The right-hand limit $\quad \lim\limits_{x \to 0_+} f(x) = \lim\limits_{x \to 0_+} \sqrt{4+x^2} = 2$

Thus, exists $\quad \lim\limits_{x \to 0} f(x) = 2.$

Since **f(0)=2;** $\quad \lim\limits_{x \to 0} \mathbf{f(x)=2;} \quad \lim\limits_{x \to 0} \mathbf{f(x)=2=f(0)}$

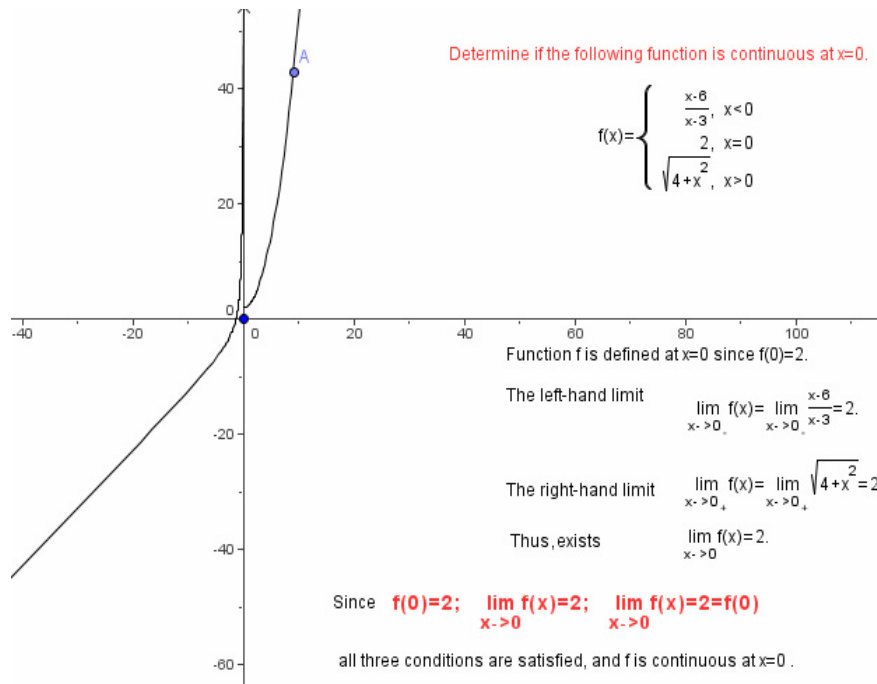all three conditions are satisfied, and f is continuous at x=0 .

Fig. 4.

If a function is not continuous at a point of its domain, it is said that it has a discontinuity at that point. Next we can consider the following questions: Why is not a function continuous? What do the graphs of discontinuous functions look like? What types of discontinuities exist? Can a discontinuity be removed? What is the connection between continuity and differentiability? ...

3. Exploring the Effects of Using GeoGebra in Calculus

3.1. Didactic Experiment in Calculus Teaching

Calculus is the important section, taught at the subject Mathematics II, during the second semester at The Accredited Business-Technical School of the vocational studies in Uzice, Serbia. The research entitled "How an innovative pedagogical approach and the potential of the GeoGebra could be use in the teaching and learning some part of calculus?" was realized in the summer term of the school academic year 2008/2009, on the subject Mathematics II.

3.2. Global Outcomes

To evaluate whether there is a significant difference in student's math achievement before and after GeoGebra-based math instruction into some important concepts of calculus.

3.3. Methodology

3.3.1. Subjects

This experiment used a sample of total 31 (gender: 19 female, 12 male) students of The Accredited Business-Technical School of the vocational studies in Uzice, Serbia, which was randomly choosing and taught by the same teacher.

3.3.2. Instruments

To achieve the purpose of the experiment, the researcher has adopted two instruments, including:
− Specially designed GeoGebra applets. As alternative representations, this experiment used graphic visualizations, obtained from GeoGebra, as a complement to already existing representations of calculus, which is normally being used in regular teaching.
− An equivalent pretest-posttest design of test (Cronbach's Alpha reliability was 0.784).

Ljubica Diković

3.4.   Procedure

3.4.1. Teaching Methodology for Experimental Group

After lectures and exercises on Calculus had been conducted in traditional way, the experimental group was included in the block teaching of Calculus, which was created by GeoGebra. The experimental group worked in the computer laboratory. Their work could be described as "mathematical experimenting". The students in the experimental group were free to communicate among each other and they developed group discussion. Sometimes, the teacher coordinated, encouraged and supported some comments in crucial points. The teacher was also available to each student, on his or her request. So, the teaching method was supplemented by the group work, individual research and investigations.

3.5.   Evaluation of the Experiment

The evaluation was conducted internally (by teaching staff). Pre-test was administered on the beginning of the experiment, post-test were administered at the end of the experiment.

3.6.   Quantitative Data Analysis

3.6.1. Hypothesis

Null $H_0(\mu_1 = \mu_2)$: There is no significant difference between the means of the two variables, before the experimental procedure (pre-test) and afterwards (post-test).

Alternate $H_1(\mu_1 \neq \mu_2)$: There is a significant difference between the means of the two variables.

3.6.2. Statistical Results

The paired-samples t-test revealed a significant differences in scores before and after the GeoGebra's workshops, t (30) = -8.59, p < .02. This indicates that the mean score after the workshops (M = 51.64) was significantly higher than the mean before the workshops (M = 22.95).

3.7.   Discussion and Significance of the Experiment

The aim of this research was to try to check, on the basis of the scores the students obtained on a test, if there was a positive effect of using GeoGebra applets in the differential calculus teaching. The test in question consisted of ten simple tasks, chosen in order to check the elementary knowledge of students in differential calculus: what is the "accumulation point" of a sequence, computing some basic limits of the functions, computing left-hand and right-hand limit of the functions, understanding the Δy of the function on a interval, understanding the instantaneous rate of change of function at a point, geometrical interpretation of derivative, etc.

Since the statistical analysis of paired-samples t-test showed that the scores at the repeated test were significantly better (t (30) = -8.59, p < .02), it confirmed the fact that the use of the applets created with the help of GeoGebra and used in differential calculus teaching, had a positive effect on the understanding and knowledge of the students.

That further shows that GeoGebra can be a powerful tool for visualization and stimulation of the key notions of differential calculus (the slope of tangent line, connection between slope of the tangent line and graph of the gradient function, continuity/discontinuity of function, connection between differentiability and continuity etc.); the fact that helped the students improve their knowledge.

The deficiency of the research can be the sample size. That is why our further research should be directed to statistical analysis of parallel samples made of a group of examinees (groups of related faculties or groups of different faculties) so as to confirm the results of the research in a greater extent.

4.   Conclusion

In this paper, I highlighted some opportunities and examples on how GeoGebra can be used in classrooms to explore some basic concepts in linear algebra and calculus. It was showed that GeoGebra has many possibilities to help students to get an intuitive feeling and to visualize adequate math process. The use of this software's tools allows students to explore a wider range of function types, and provides students to make the connections between symbolic and visual representations.

5.   References

1. Hohenwarter, M., Hohenwarter, J., Kreis, Y., & Lavicza, Z. (2008). Teaching and learning calculus with free dynamic mathematics software GeoGebra, 11th

Ljubica Diković

International Congress on Mathematical Education. Monterrey, Nuevo Leon, Mexico

2. Wurnig O. (2008), Some Problem Solving examples of Multiple solutions using cas and dgs, Proceedings of the Discussing Group 9 : Promoting Creativity for All Students in Mathematics Education, The 11th International Congress on Mathematical Education, Monterrey, Mexico.

3. Herceg Dj., Herceg D.(2007), Numerics with GeoGebra in high school, First Central- and Eastern European Conference on Computer Algebra- and Dynamic Geometry Systems in Mathematics Education, University of Pécs, Pollack Mihály Faculty of Engineering, Hungary.

4. Dikovich Lj.(2007), „An Interactive Learning and Teaching of Linear Algebra by Web Technologies: Some Examples", Journal the Teaching of mathematics, Publisher: The Mathematical Society of Serbia, Beograd, ISSN: 1451-4966, Issue: X_2, Pages: 109 – 116.

5. Hohenwarter, M.; Preiner, J. (2007): Dynamic Mathematics with GeoGebra. Journal for Online Mathematics and its Applications, Volume 7, Article ID 1448.

6. Hohenwarter, M.; Preiner, J., Taeil Yi, (2007): Incorporating GeoGebra into Teaching Mathematics at the College Level, Proceedings of ICTCM 2007, Boston, MA, available on address http://www.geogebra.org/publications/ 2007_ICTCM_geogebra/

7. Hohenwarter, M., & Lavicza, Z. (2007). Mathematics teacher development with ICT: towards an International GeoGebra Institute. In D. Küchemann (Ed.), Proceedings of the British Society for Research into Learning Mathematics. 27(3). University of Northampton, UK: BSRLM.

8. K. Ruthven, S. Hennessy, S. Brindley (2004), Teacher representations of the successful use of computer-based tools and resources in secondary-school English, mathematics and science, Teaching and Teacher Education, Volume 20, Issue 3, pages 259-275.

9. Blubaugh, W. (2004), Teaching and Learning Business Calculus Through Technology, Electronic Proceedings of the Seventeenth Annual International Conference on Technology in Collegiate Mathematics New Orleans, Louisiana, ISBN 0-321-43792-6, Paper S052, 2004

10. Waxman H., Connell M.,, Gray J. (2002), A Quantitative Synthesis of Recent Research on the Effects of Teaching and Learning With Technology on Student Outcomes, NCREL, available on address www.coe.ufl.edu/Courses/eme5054/ Foundations/Articles/waxman.pdf

11. NCTM (2000). Principles and Standards for School Mathematics. http://standards.nctm.org

12. van Joolingen, Wouter (1999): Cognitive tools for discovery learning. International Journal of Artificial Intelligence in Education, 10, p. 385-397.

13. http://www.learning-theories.com/discovery-learning-bruner.html

14. http://opensource.org/

15. http://www.dwheeler.com/oss_fs_why.html

16. http://www.gnu.org/philosophy/categories.html

17. http://www.geogebra.org

18. http://www.math.umn.edu/~rogness/mathlets/xSquaredSin1overX/

19. Little C., Interactive geometry in the classroom: old barriers and new opportunities, Proceedings of the British Society for Research into Learning Mathematics 28(2) June 2008, available at bsrlm.org.uk © the author – 49

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